Generalized maximum entropy approach to quasi-stationary states in long range systems: the example of the Hamiltonian Mean Field model

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INTRODUCTION

Systems with long-range interactions display a short-time relaxation towards quasistationary states (QSS) whose lifetime increases with the system size. In the paradigmatic Hamiltonian mean-field model (HMF) out-of-equilibrium phase transitions are predicted and numerically detected which separate homogeneous (zero magnetization) and inhomogeneous (nonzero magnetization) QSS. In the former regime, the velocity distribution presents two large, symmetric, bumps, which cannot be self-consistently explained by resorting to the conventional Lynden-Bell maximum entropy approach. To improve the theory in this respect and eventually fill the gap with the observation, we here propose a generalized maximum entropy scheme which accounts for the pseudo-conservation of a additional charges, the even momenta of the single particle distribution. These latter are set to the asymptotic values, as estimated by direct integration of the underlying Vlasov equation, which formally holds in the thermodynamic limit. Methodologically, we operate in the framework of generalized Gibbs ensemble, as sometimes defined in statistical quantum mechanics, which contains an infinite number of conserved charges. The agreement between theory and simulations is extremely satisfying, both above and below the out of equilibrium transition threshold. The fine details of the velocity profile are adequately captured upon truncation at the tenth order in the hierarchy of pseudo-conserved momenta.

THE MODEL

The HMF model describes the motion of N coupled rotators and is characterized by the following Hamiltonian:

$$H = \frac{1}{2}\sum_{j=1}^{N} p_j^2 + \frac{1}{2N}\sum_{i,j=1}^{N}\left[1-\cos(\theta_j-\theta_i)\right]$$

θ_j = orientation of the j-th rotor
p_j = conjugate momentum of the j-th rotor

To monitor the evolution of the system, it is customary to introduce the magnetization, a global order parameter defined as:

$$M = \left|\sum \mathbf{m}_i\right| \Big/ N \qquad \mathbf{m}_i = (\cos\theta_i, \sin\theta_i)$$

Starting from out of equilibrium initial conditions, the system gets trapped in Quasi-Stationary States (QSS), whose lifetime diverges when increasing the number of particles N. Importantly, when performing the mean-field limit ($N \to \infty$) before the infinite time limit, the system cannot relax towards Boltzmann-Gibbs equilibrium and remains permanently confined in QSS. In this regime, the magnetization is lower than the one predicted by the Boltzmann-Gibbs equilibrium and the system apparently displays a number of intriguing anomalies, e.g. non Gaussian velocity distributions and non standard diffusion in angle. Moreover in the $N \to \infty$ limit, the N-particle dynamics is described by the Vlasov equation:

$$\frac{\partial f}{\partial t} + p\frac{\partial f}{\partial \theta} - \frac{dV}{d\theta}\frac{\partial f}{\partial p} = 0$$

where $f(\theta,p,t)$ is the microscopic one-particle distribution function

THE GENERALIZED MAXIMUM ENTROPY SCHEME WITH PSEUDO-CONSERVED QUANTITIES

Entropy

$$s(f) = -\iint dp\,d\theta\left[\frac{f}{f_0}\ln\frac{f}{f_0} + \left(1-\frac{f}{f_0}\right)\ln\left(1-\frac{f}{f_0}\right)\right]$$

$$\int_{-\pi}^{+\pi}\int_{-\infty}^{+\infty} f(\theta,p)\,d\theta\,dp = 1$$

$$\int_{-\pi}^{+\pi}\int_{-\infty}^{+\infty} \frac{p^2}{2} f(\theta,p)\,d\theta\,dp - \frac{m_x^2-1}{2} = U$$

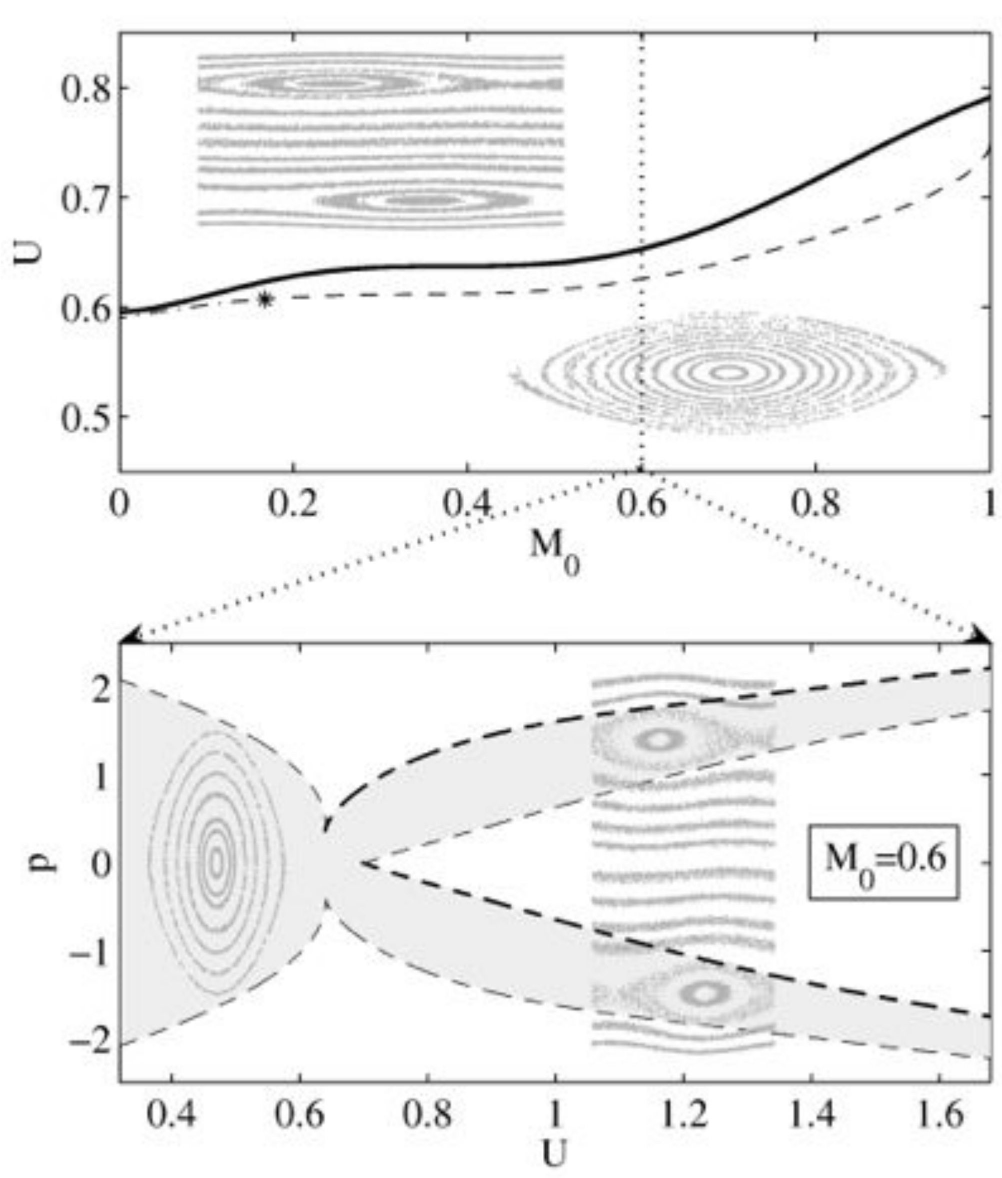
$$\int_{-\pi}^{+\pi}\int_{-\infty}^{+\infty} \cos\theta\cdot f(\theta,p)\,d\theta\,dp = m_x$$

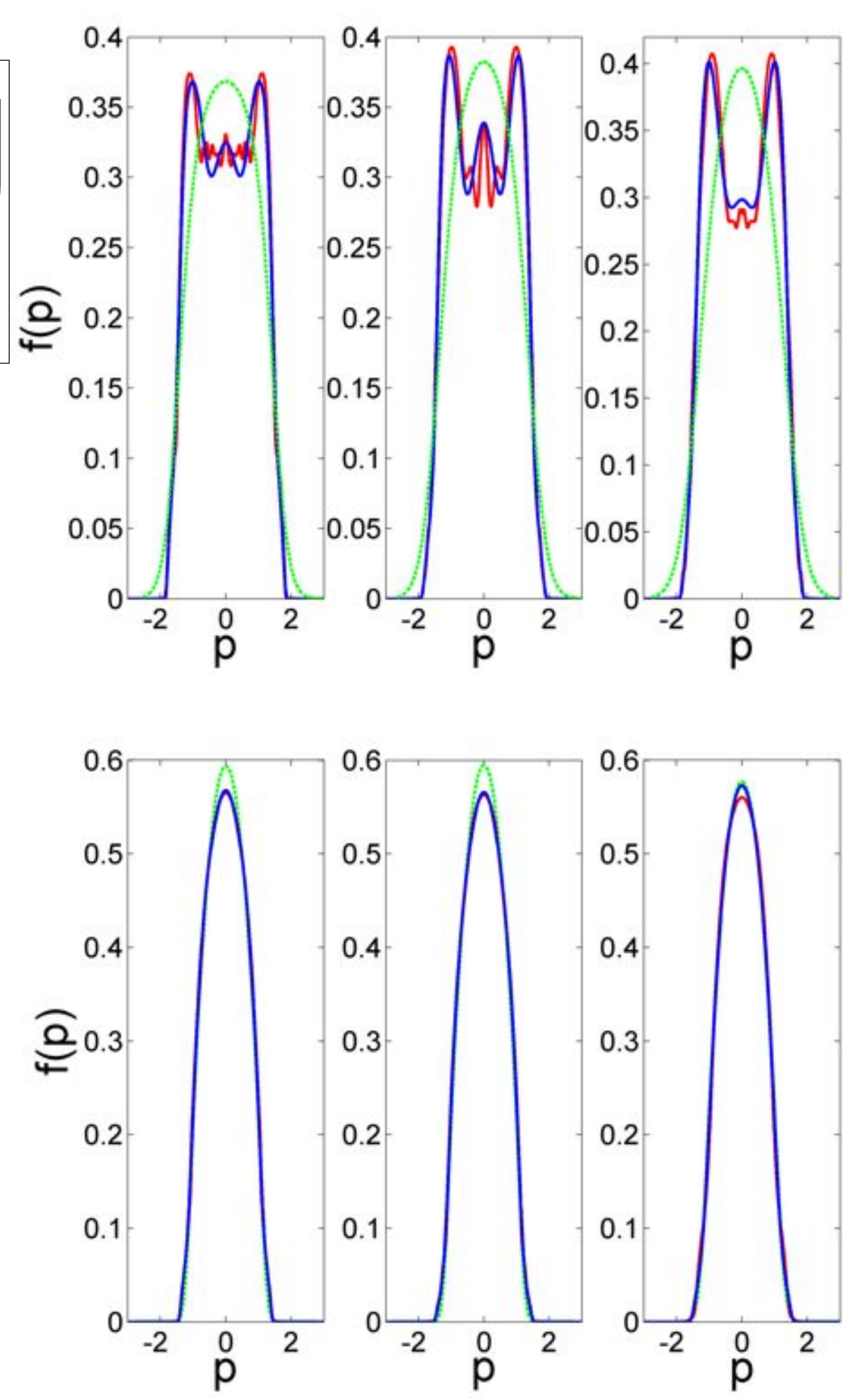
$$\varepsilon_k = \int_{-\pi}^{+\pi}\int_{-\infty}^{+\infty} p^{2k+2} f(\theta,p)\,d\theta\,dp$$

Upon maximization

$$f_k(\theta,p) = \exp\left(-\beta\frac{p^2}{2} + \beta m_x\cos\theta - \sum_{i=1}^{k}\lambda_i p^{2k+2}\right)$$

$$f = f_0\frac{f_k}{1+f_k} \qquad k = 1,2,3\ldots,n$$





The Fermionic Lynden-Bell (LB) entropy is maximised, subject to the constraints of the dynamics, to characterize the QSS. Under the standard LB paradigm, an out-of-equilibrium phase transition is found which separates between homogeneous and non-homogeneous regimes. The velocity distribution function as predicted by the LB theory (dashed green) fails to capture the fine details of the profile, as recorded via direct Vlasov simulations (solid red). A generalized theory which sets the even average momenta to the asymptotic values determined by Vlasov dynamics, after the initial fast relaxation, proves definitely more adequate in explaining the observations (solid blue). This holds both above and below the phase transition threshold.